

Hydrodynamic fluctuations and the minimum shear viscosity of the dilute Fermi gas at unitarity

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Abstract

We study hydrodynamic fluctuations in a non-relativistic fluid. We show that in three dimensions fluctuations lead to a minimum in the shear viscosity to density ratio η/n as a function of the temperature. The minimum provides a bound on η/n which is independent of the conjectured bound in string theory, $\eta/s \geq \hbar/(4\pi k_B)$, where s is the entropy density. For the dilute Fermi gas at unitarity we find $\eta/n \gtrsim 0.3\hbar$. This bound is not rigorous, but it can only be avoided if the domain of validity of hydrodynamics is anomalously small. We also find that the viscous relaxation time of a hydrodynamic mode with frequency ω diverges as $1/\sqrt{\omega}$, and that the shear viscosity in two dimensions diverges as $\log(1/\omega)$.

I. INTRODUCTION

It is now widely appreciated that fluid dynamics can be viewed as an effective long-distance theory for a classical or quantum many-body system at non-zero temperature. Effective theories make systematic predictions for correlation functions order-by-order in a low-momentum expansion. These predictions depend on a small number of microscopic parameters. In the case of fluid dynamics the microscopic parameters are the equation of state and the transport coefficients.

Effective (field) theories are constructed using the following procedure: i) Identify the low energy degrees of freedoms. ii) Write down the most general local effective action consistent with the symmetries of the problem. This action is typically expressed in terms of low energy fields and their derivatives. The coefficients of allowed terms in the effective action are free parameters called low energy constants. iii) Determine what terms in the effective action have to be included in order to compute a correlation function to a given order in the low energy expansion. This is known as the “power counting”. Typically, the leading contribution arises from tree level diagrams involving operators with the minimal number of derivatives, and higher order corrections arise both from higher derivative operators, and from loop diagrams with the leading operators. In some cases diagrams may have to be summed to all orders. For example, the sum of all tree diagrams corresponds to solving a classical field equation.

In fluid dynamics the low energy modes are fluctuations of the conserved charges. In the case of a one-component non-relativistic fluid the conserved charges are the particle density, the energy density, and the momentum density. The derivative expansion is implemented at the level of the constitutive equations. This means that the conserved currents are expressed in terms of derivatives of the thermodynamic variables. The effective theory without derivative terms in the currents is called ideal fluid dynamics, and the equation of motion at one-derivative order is known as the Navier-Stokes equation. The validity of fluid dynamics requires that derivative corrections to the currents are small. This condition does not preclude the possibility that small corrections can exponentiate as one solves the equations of motion. Solutions of the Navier-Stokes equation are qualitatively different from solutions of ideal fluid dynamics. In ideal fluid dynamics the motion is time reversible, sound modes are not damped, and diffusive modes do not exist. This implies that in most cases

one has to retain at least one-derivative terms in the constitutive equations. Two derivative terms have also been studied [1, 2], but the corrections are typically small. In relativistic fluid dynamics two-derivative terms improve the stability of equations of motion, and second order terms are now routinely included in hydrodynamic simulations of relativistic heavy ion collisions [3].

Loop corrections in fluid dynamics arise from thermal fluctuations. Fluctuations are known to be important in the vicinity of second order phase transitions [4], but they are rarely considered in the case of non-critical fluids. In this work we will consider the contribution of fluctuations to the correlation function of the stress tensor in a simple non-relativistic fluid. We study the implications of our results for the shear viscosity of the unitary Fermi gas. The unitary Fermi is known to have a very low viscosity [5–8], close to the value obtained from the AdS/CFT (Anti-deSitter Space/Conformal Field Theory) correspondence, $\eta/s = \hbar/(4\pi k_B)$ [9, 10]. Here, η is the shear viscosity and s is the entropy density, \hbar is Planck’s constant and k_B is Boltzmann’s constant. We will set $\hbar = k_B = 1$ in the following. We will show that the small shear viscosity enhances the role of fluctuations, but we also show that fluctuations imply a lower limit on how small the viscosity can get. We demonstrate that fluctuations lead to a non-analytic term in the viscous relaxation time in three spatial dimensions, and to a logarithmic divergence of the shear viscosity in two dimensions. Finally, we discuss the possibility of observing these non-analytic terms in experiments with trapped atomic gases.

Our work builds on a substantial literature related to fluctuations in fluid dynamics, beginning with the work of Landau [11]. The role of fluctuations in critical transport phenomena was summarized in the review article by Hohenberg and Halperin [4] and the text books by Ma and Onuki [12, 13]. Diagrammatic methods are discussed by a number of authors, for example in [14–16]. Our work closely follows a recent study of fluctuations in relativistic fluids, see [17, 18], the recent review [19], and the related work in [20].

II. KUBO FORMULA

In this section we will determine the low energy behavior of the retarded correlation function of the stress tensor using the equation of fluid dynamics at next-to-leading order in the gradient expansion. This result can be used to derive the standard Kubo formula for the

shear viscosity, as well as a new Kubo formula for the viscous relaxation time. We will employ the formalism developed in [2, 21–23], which is based on coupling the theory to a non-trivial background metric $g_{ij}(t, \vec{x})$. Correlation functions of the stress tensor can be computed by using linear response theory, and the requirements of Gallilean and conformal symmetry can be incorporated by requiring the equations of fluid dynamics to satisfy diffeomorphism and conformal invariance.

The retarded correlation function of the stress tensor Π^{ij} is defined by

$$G_R^{ijkl}(\omega, \mathbf{k}) = -i \int dt \int d\mathbf{x} e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \Theta(t) \langle [\Pi^{ij}(t, \mathbf{x}), \Pi^{kl}(0, \mathbf{0})] \rangle. \quad (1)$$

G_R determines the stresses induced by a small perturbation $g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}(t, \mathbf{x})$ around the flat metric. We have

$$\delta \Pi^{ij} = -\frac{1}{2} G_R^{ijkl} h_{kl}. \quad (2)$$

In fluid dynamics we expand the stress tensor in derivatives of the local thermodynamic variables P, ρ, \mathbf{v} , where P is the pressure, ρ is the density, and \mathbf{v} is the fluid velocity. We write $\Pi_{ij} = \Pi_{ij}^0 + \delta \Pi_{ij}$, where

$$\Pi_{ij}^0 = \rho v_i v_j + P g_{ij} \quad (3)$$

is the ideal fluid part, and $\delta \Pi_{ij}$ is the viscous correction. In a conformally invariant fluid the leading term is $\delta \Pi_{ij} = -\eta \sigma_{ij}$ with

$$\sigma_{ij} = \nabla_i v_j + \nabla_j v_i + \dot{g}_{ij} - \frac{2}{3} g_{ij} \langle \sigma \rangle, \quad (4)$$

$$\langle \sigma \rangle = \nabla \cdot v + \frac{\dot{g}}{2g}, \quad (5)$$

where σ_{ij} is the shear stress tensor, η is the shear viscosity, $g_{ij} \langle \sigma \rangle$ is the bulk stress tensor, and ∇_i is the covariant derivative associated with g_{ij} . Note that the bulk viscosity of a conformal fluid is zero. In [2] we classified all terms up to second order in derivatives. We have

$$\begin{aligned} \delta \Pi_{ij} = & -\eta \sigma_{ij} + \eta \tau_R \left(g_{ik} \dot{\sigma}_j^k + v^k \nabla_k \sigma_{ij} + \frac{2}{3} \langle \sigma \rangle \sigma_{ij} \right) + \lambda_1 \sigma_{\langle i}^k \sigma_{j \rangle k} + \lambda_2 \sigma_{\langle i}^k \Omega_{j \rangle k} \\ & + \lambda_3 \Omega_{\langle i}^k \Omega_{j \rangle k} + \gamma_1 \nabla_{\langle i} T \nabla_{j \rangle} T + \gamma_2 \nabla_{\langle i} P \nabla_{j \rangle} P + \gamma_3 \nabla_{\langle i} T \nabla_{j \rangle} P \\ & + \gamma_4 \nabla_{\langle i} \nabla_{j \rangle} T + \gamma_5 \nabla_{\langle i} \nabla_{j \rangle} P + \kappa_R R_{\langle ij \rangle}, \end{aligned} \quad (6)$$

where τ_R is the viscous relaxation time, $\lambda_i, \gamma_i, \kappa_R$ are second order transport coefficients, $\Omega_{ij} = \nabla_i v_j - \nabla_j v_i$ is the vorticity tensor, T is the temperature, and R_{ij} is the Ricci tensor

associated with g_{ij} . Note that R_{ij} vanishes in flat space $g_{ij} = \delta_{ij}$, but keeping terms involving the curvature is crucial for obtaining the correct low energy expansion of G_R .

We will concentrate on the “pure shear” component G_R^{xyxy} . For this purpose we consider a perturbation of the form $h_{xy}(z, t)$. From the linearized Euler equation we can see that the perturbation does not induce a shift in the density, temperature, or velocity. This means that we can directly read off $\delta\Pi_{ij}$ from equ. (6). We find

$$G_R^{xyxy}(\omega, k) = P - i\eta\omega + \tau_R\eta\omega^2 - \frac{\kappa_R}{2}k^2 + O(\omega^3, \omega k^2), \quad (7)$$

which implies the familiar Kubo relation for the shear viscosity

$$\eta = -\lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{d}{d\omega} \text{Im} G_R^{xyxy}(\omega, \mathbf{k}) \quad (8)$$

as well as a new Kubo formula for the viscous relaxation time

$$\tau_R\eta = \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{1}{2} \frac{d^2}{d\omega^2} \text{Re} G_R^{xyxy}(\omega, \mathbf{k}). \quad (9)$$

This result is simpler than the corresponding formula in relativistic hydrodynamics [24], which also involves a term proportional to κ_R .

III. HYDRODYNAMIC FLUCTUATIONS

In this section we will study the contribution of fluctuations to the retarded correlation function. For this purpose it is convenient to start from the symmetrized correlation function

$$G_S^{xyxy}(\omega, \mathbf{k}) = \int d^3x \int dt e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \left\langle \frac{1}{2} \{ \Pi_{xy}(t, \mathbf{x}), \Pi_{xy}(0, 0) \} \right\rangle. \quad (10)$$

This function is related to the retarded correlator by the fluctuation dissipation theorem.

For $\omega \rightarrow 0$ we have

$$G_S(\omega, \mathbf{k}) \simeq -\frac{2T}{\omega} \text{Im} G_R(\omega, \mathbf{k}). \quad (11)$$

In the low frequency, low momentum limit we can use the form of the stress tensor in fluid dynamics, $\Pi_{xy} = \rho v_x v_y - \eta(\nabla_x v_y + \nabla_y v_x) + O(\nabla^2)$, and expand the hydrodynamic variables around their mean values, $\rho = \rho_0 + \delta\rho$ etc. We will use the Gaussian approximation and write expectation values of products of fluctuating fields as products of two point functions. The ideal (zero derivative) terms in the stress tensor give one and two loop graphs involving velocity-velocity and density-density correlation functions. We will show in the appendix

that additional loops are suppressed by powers of $1/(n^{1/3}l_{mfp})$, where n is the number density, and l_{mfp} is the mean free path. We will therefore concentrate on the one-loop graph

$$G_S^{xyxy}(\omega, 0) = \rho_0^2 \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\Delta_S^{xy}(\omega', \mathbf{k}) \Delta_S^{yx}(\omega - \omega', \mathbf{k}) + \Delta_S^{xx}(\omega', \mathbf{k}) \Delta_S^{yy}(\omega - \omega', \mathbf{k})] \quad (12)$$

where Δ_S^{ij} is the symmetrized velocity correlation function

$$\Delta_S^{ij}(\omega, \mathbf{k}) = \int d^3x \int dt e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \left\langle \frac{1}{2} \{v^i(t, \mathbf{x}), v^j(0, 0)\} \right\rangle. \quad (13)$$

We are ultimately interested in the retarded, not the symmetrized, correlation function. At low frequency the retarded function can be written as

$$G_R^{xyxy}(\omega, 0) = \rho_0^2 \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\Delta_R^{xy}(\omega', \mathbf{k}) \Delta_S^{yx}(\omega - \omega', \mathbf{k}) + \Delta_S^{xy}(\omega', \mathbf{k}) \Delta_R^{yx}(\omega - \omega', \mathbf{k}) + \Delta_R^{xx}(\omega', \mathbf{k}) \Delta_S^{yy}(\omega - \omega', \mathbf{k}) + \Delta_S^{xx}(\omega', \mathbf{k}) \Delta_R^{yy}(\omega - \omega', \mathbf{k})], \quad (14)$$

where we have used the fluctuation dissipation relation (11). This relation generalizes: retarded correlation functions of hydrodynamic variables have diagrammatic expansions in terms of retarded and symmetrized correlation functions [4, 12–16, 19].

The velocity correlation function can be decomposed into longitudinal and transverse parts

$$\Delta_{S,R}^{ij}(\omega, \mathbf{k}) = (\delta^{ij} - \hat{\mathbf{k}}^i \hat{\mathbf{k}}^j) \Delta_{S,R}^T(\omega, \mathbf{k}) + \hat{\mathbf{k}}^i \hat{\mathbf{k}}^j \Delta_{S,R}^L(\omega, \mathbf{k}). \quad (15)$$

The transverse part is purely diffusive. The symmetrized correlation function is [11]

$$\Delta_S^T(\omega, \mathbf{k}) = \frac{2T}{\rho} \frac{D_\eta k^2}{\omega^2 + (D_\eta k^2)^2}, \quad (16)$$

where $k = |\mathbf{k}|$ and $D_\eta = \eta/\rho$ is the momentum diffusion constant, also known as the kinetic viscosity. The retarded correlation function is given by

$$\Delta_R^T(\omega, \mathbf{k}) = \frac{1}{\rho} \frac{-D_\eta k^2}{-i\omega + D_\eta k^2}. \quad (17)$$

The longitudinal correlation function can be reconstructed from the density-density correlation function (the dynamic structure factor) using current conservation, $i\omega\delta\rho = i\rho\mathbf{k} \cdot \mathbf{v}_L$, where $\mathbf{v} = \mathbf{v}_L + \mathbf{v}_T$ with $\mathbf{k} \cdot \mathbf{v}_T = 0$ and $\mathbf{k} \times \mathbf{v}_L = 0$. We find

$$\Delta_S^L(\omega, \mathbf{k}) = \frac{2T}{\rho} \left\{ \frac{\Gamma\omega k^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma\omega k^2)^2} + \left(\frac{c_p}{c_v} - 1\right) \frac{1}{c_s^2} \frac{D_T \omega^2}{\omega^2 + (D_T k^2)^2} - \left(\frac{c_p}{c_v} - 1\right) \frac{1}{c_s^2} \frac{(\omega^2 - c_s^2 k^2) D_T \omega^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma\omega k^2)^2} \right\}. \quad (18)$$

The first two terms have a clear physical interpretation as the contributions from propagating sound waves and diffusive heat transport. The third term is required to satisfy sum rules. This term is suppressed near the sound pole $\omega^2 \simeq c_s^2 k^2$. In equ. (18) c_s is the speed of sound, Γ is the sound attenuation constant, $D_T = \kappa/(c_p \rho)$ is the thermal diffusion constant, κ is the thermal conductivity, c_p is the specific heat per unit mass at constant pressure and c_v is the specific heat at constant volume. The sound attenuation constant is

$$\Gamma = \frac{4\eta}{3\rho} + \frac{\zeta}{\rho} + \frac{\kappa}{\rho} \left(\frac{1}{c_v} - \frac{1}{c_p} \right) = \frac{4\eta}{3\rho} \left[1 + \frac{3\zeta}{4\eta} + \frac{3\Delta c_p}{4Pr} \right], \quad (19)$$

where $\Delta c_p = (c_p - c_v)/c_v$ and $Pr = (c_p \eta)/\kappa$ is the Prandtl number, the ratio of momentum to thermal diffusion. At high temperature $\Delta c_p = 2/3$ and $Pr = 2/3$ [25], and at low temperature $\Delta c_p/Pr \rightarrow 0$. In the case of a conformal fluid the bulk viscosity vanishes, $\zeta = 0$. This implies that $\Gamma = \frac{7}{3}D_\eta$ at high temperature, and $\Gamma = \frac{4}{3}D_\eta$ at low temperature.

At low frequency and momentum the symmetrized correlation function can be further simplified. We illustrate the result in the case of the sound pole. We can write

$$\Delta_S^{sound}(\omega, \mathbf{k}) \simeq \frac{\Gamma T k^2}{2\rho} \left\{ \frac{1}{(\omega - c_s k)^2 + \left(\frac{\Gamma k^2}{2}\right)^2} + \frac{1}{(\omega + c_s k)^2 + \left(\frac{\Gamma k^2}{2}\right)^2} \right\}, \quad (20)$$

which is correct up to terms of order $\Gamma k/c_s$. The retarded correlator is

$$\Delta_R^{sound}(\omega, \mathbf{k}) \simeq \frac{\omega}{2\rho} \left\{ \frac{1}{\omega - c_s k + i\frac{\Gamma k^2}{2}} + \frac{1}{\omega + c_s k + i\frac{\Gamma k^2}{2}} \right\}, \quad (21)$$

and an analogous expression holds for the sum rule term in equ. (18).

We can insert the decomposition of the velocity correlation function given in equ. (15) into the one-loop result for the retarded correlation function, equ. (14). This gives a number of terms, corresponding to the contribution of shear modes, sound modes, a mixed shear-sound contribution, and the contribution of diffusive heat modes, see Fig. 1. We discuss these contributions in turn:

1. Shear modes: This is the contribution which is easiest to compute. The frequency integral can be done by contour integration. We find

$$G_R^{xyxy}(\omega, 0)|_{shear} = -\frac{7T}{30\pi^2} \int dk \frac{k^4}{k^2 - i\omega/(2D_\eta)}. \quad (22)$$

This integral is UV divergent. We regulate the divergence by introducing a momentum cutoff Λ . We then expand the integral in the low frequency regime. We get

$$G_R^{xyxy}(\omega, 0)|_{shear} = -\frac{7}{90\pi^2} T \Lambda^3 - i\omega \frac{7T\Lambda}{60\pi^2 D_\eta} + (1+i)\omega^{3/2} \frac{7T}{240\pi D_\eta^{3/2}} + O(\omega^{5/2}). \quad (23)$$

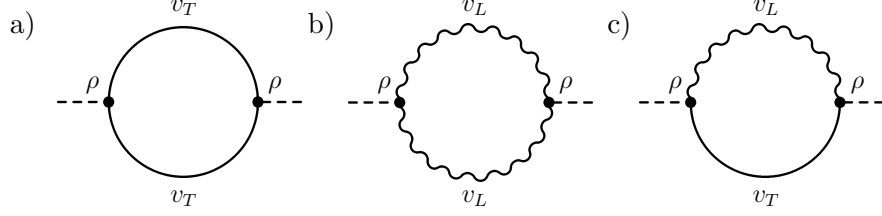


FIG. 1: Diagrammatic representation of the leading contribution of thermal fluctuations to the stress tensor correlation function. Solid lines labeled v_T denote the transverse velocity correlator, dominated by the shear pole, and wavy lines labeled v_L denote the longitudinal velocity correlator, governed by the sound pole and the diffusive heat mode.

The physical meaning of these terms can be understood by comparing with the Kubo relation in equ. (7). The first term is a fluctuation contribution to the pressure, and the second term is a correction to the shear viscosity. The imaginary part of the third term can be viewed as a frequency dependent correction to the shear viscosity, and the real part is a frequency dependent contribution to the relaxation time which diverges as $\omega^{-1/2}$ in the low frequency limit. The existence of this term is sometimes interpreted as an indication that hydrodynamics breaks down beyond the Navier-Stokes (one-derivative) order.

2. Sound modes: In order to calculate the contribution from sound modes we use equ. (20) and (21). This leads to two types of terms, depending on whether the real parts of the poles of the propagators in the ω -plane have the same or opposite sign. The contribution from terms with opposite real parts has the same structure as the shear mode term. We get

$$G_R^{xyxy}(\omega, 0)|_{sound} = \frac{1}{90\pi^2} T \Lambda^3 - i\omega \frac{T \Lambda}{30\pi^2 \Gamma} + (1 + i)\omega^{3/2} \frac{\sqrt{2}T}{120\pi \Gamma^{3/2}} + O(\omega^{5/2}). \quad (24)$$

The contribution from terms with real parts of the same sign is not infrared sensitive and does not contribute to the retarded correlation function at $O(\omega)$ or $O(\omega^{3/2})$.

3. Shear-sound contribution: The shear-sound contribution has the structure

$$G_R^{xyxy}(\omega, 0)|_{sh-so} \sim \int_{-\Lambda}^{\Lambda} dk \frac{k^4}{k^2 + i(\omega - c_s k)/D_s}, \quad (25)$$

where $D_s = D_\eta + \Gamma/2$ and the range of the k -integral is $[-\Lambda, \Lambda]$ because of the $k \leftrightarrow -k$ symmetry of the sound propagator in equ. (20). We get

$$G_R^{xyxy}(\omega, 0)|_{sh-so} \sim \frac{D_s^2 \Lambda^5}{c_s^2} + i\omega \frac{D_s \Lambda^3}{c_s^2} + O(\omega^2), \quad (26)$$

which is suppressed relative to the pure shear and sound contributions by a factor $D_s\Lambda/c_s \ll 1$ (see Sec. IV). We also note that the mixed shear-sound term does not give non-analytic terms of the form $\omega^{3/2}$.

4. Diffusive heat modes: The contribution of diffusive heat modes is very similar to the shear term, but the residue of the heat mode is proportional to ω^2/c_s^2 instead of k^2 . In the diffusive regime $\omega^2 \ll c_s^2 k^2$. We find

$$G_R^{xyxy}(\omega, 0)|_{heat} \sim \frac{\omega^2 D_T^2}{c_s^4} \int dk \frac{k^2}{k^2 - i\omega/(2D_T)} \sim \frac{\omega^2 D_T^2 \Lambda^3}{c_s^4}, \quad (27)$$

which is much smaller than the shear term.

We conclude that the main contribution arises from the pure shear and sound terms. We will combine these two contributions using the approximation $\Gamma \simeq \frac{4}{3}D_\eta$, which corresponds to the low temperature regime. This is the more interesting regime because D_η is small and the role of fluctuations is enhanced. We find

$$G_R^{xyxy}(\omega, 0)|_{tot} = const - i\omega \frac{17T\Lambda}{120\pi^2 D_\eta} + (1+i)\omega^{3/2}T \frac{7 + \left(\frac{3}{2}\right)^{3/2}}{240\pi D_\eta^{3/2}} + O(\omega^2). \quad (28)$$

As noted above the $i\omega$ term is a contribution to the shear viscosity. This term is cutoff dependent, but the physical viscosity must be independent of an arbitrary cutoff. This implies that the bare viscosity must be cutoff dependent too, and that the cutoff dependence of the bare viscosity is governed by a renormalization group equation. It is important that the non-analytic $\omega^{3/2}$ term is not cutoff dependent, because any cutoff dependence in this contribution cannot be absorbed into the parameters of the hydrodynamic description.

IV. PHENOMENOLOGICAL ESTIMATES

In this section we study phenomenological implications of the results derived in the previous section. We observed that $i\omega$ term in the retarded correlation function can be combined with the bare shear viscosity to give a physical viscosity

$$\eta_{phys} = \eta + \frac{17}{120\pi^2} \frac{\rho T \Lambda}{\eta}. \quad (29)$$

An interesting consequence of this result is the fact that the physical viscosity cannot be arbitrary small [17], because equ. (29) has a minimum as long as the bare viscosity is positive. The bare viscosity must be positive for the hydrodynamic expansion to be well defined. The

value at the minimum depends on the value of the cutoff; the larger the cutoff the stronger the bound on η becomes. The largest possible value of the cutoff is determined by the condition that the gradient expansion on which hydrodynamics is based must be valid for all $k \lesssim \Lambda$. In the following we will study this condition separately in the shear and sound channel.

1. Shear channel: Shear modes are characterized by $\omega \sim D_\eta k^2$. Corrections arise from higher order terms in the derivative expansion. For non-zero frequency the leading correction is due to the relaxation time. We have $\omega \sim D_\eta k^2 \ll \tau_R^{-1}$. For this relation to be maintained for all $k < \Lambda$ we need to require that $\Lambda \lesssim K_{hyd}$ with $K_{hyd} = (\tau_R D_\eta)^{-1/2}$. In kinetic theory $\tau_R = \eta/P$ [2, 23, 26] and

$$K_{hyd} \simeq \frac{1}{D_\eta} \left(\frac{P}{\rho} \right)^{1/2}. \quad (30)$$

2. Sound channel: In the sound channel we have $\omega \sim c_s k \ll \Gamma k^2$. Using $\Gamma \simeq \frac{4}{3} D_\eta$ we find

$$K_{hyd} \simeq \frac{3}{4 D_\eta} \left(\frac{\partial P}{\partial \rho} \right)_s^{1/2}. \quad (31)$$

For a weakly interacting gas $(\partial P)/(\partial \rho)_s \simeq (5P)/(3\rho)$, and equ. (30) differs from equ. (31) by a factor very close to one, $\sqrt{16/15} \simeq 1.03$. In the following we will use equ. (30) as our estimate for the cutoff. We note that near a critical point the speed of sound can go to zero, and the contribution of sound waves is strongly suppressed relative to shear modes.

It is interesting to consider the microscopic meaning of the ultraviolet scale K_{hyd} . In kinetic theory $\eta \sim n \bar{p} l_{mfp}$, where $\bar{p} \sim \sqrt{mT}$ is the mean momentum, and l_{mfp} is the mean free path. For a weakly interacting gas $P \simeq nT$ and the ultraviolet scale is $K_{hyd} = (\rho/\eta)(P/\rho)^{1/2} \sim l_{mfp}^{-1}$. This is physically reasonable: It does not make sense to consider hydrodynamic fluctuations with wavelengths shorter than the mean free path.

Finally, we can estimate the minimum shear viscosity. Using equ. (29) and $\Lambda = K_{hyd}$ from equ. (30) we get

$$\left(\frac{\eta}{n} \right)_{phys} = \frac{\eta}{n} + \frac{17}{\sqrt{2} 80} \left(\frac{T}{T_F} \right)^{3/2} \left(\frac{n}{\eta} \right)^2 \left(\frac{P}{nT} \right)^{1/2}, \quad (32)$$

where we have introduced the Fermi temperature $T_F = k_F^2/(2m)$ and the Fermi momentum $k_F = (3\pi^2 n)^{1/3}$. Minimizing this expression with respect to the bare viscosity we find

$$\left(\frac{\eta}{n} \right)_{phys} \gtrsim \frac{3}{4} \left(\frac{289}{50} \right)^{1/6} \left(\frac{T}{T_F} \right)^{1/2} \left(\frac{P}{nT} \right)^{1/6} \simeq 1.005 \left(\frac{T}{T_F} \right)^{1/2} \left(\frac{P}{nT} \right)^{1/6}. \quad (33)$$

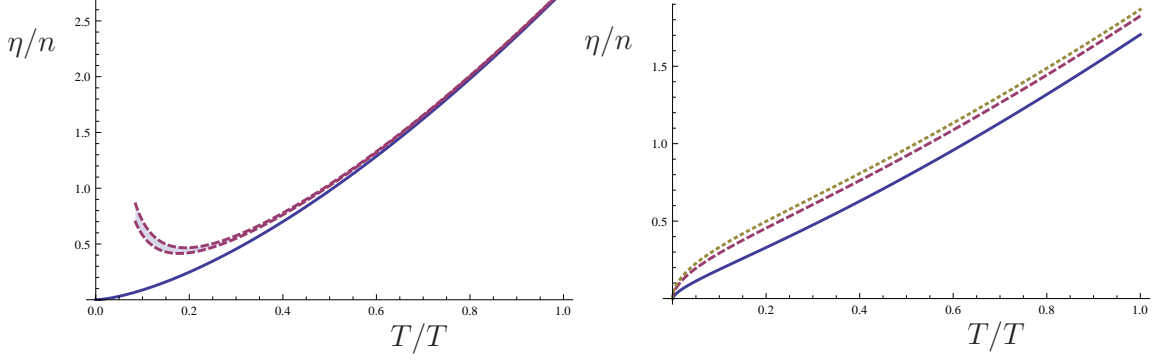


FIG. 2: Shear viscosity to density ratio η/n as a function of T/T_F , where T_F is the local Fermi temperature. The left panel shows η/n for the unitary gas in three dimensions. The solid line is the result in kinetic theory and the dashed line includes fluctuations. The band shows the uncertainty if the cutoff is varied in the regime $\Lambda = (0.5 - 1.0)K_{hyd}$. The right panel shows the two-dimensional gas at the crossover point $T_{a,2d} = T_F$. The solid line is the kinetic result. The dashed and dotted lines include fluctuations where we have used $\Lambda = K_{hyd}$ and ω was taken to be the frequency of the quadrupole mode in a harmonic trap with $N = 10^4$ and $N = 10^5$ particles.

Fluctuations imply a bound on the shear viscosity that decreases with the temperature¹. Kinetic theory calculations show that the mean free path and the bare viscosity become very large in the limit $T \ll T_c$ [27, 28]. It is therefore most interesting to consider the implications of the bound near the critical temperature T_c . In this regime $[P/(nT)]^{1/6} \simeq 1$. Using $T_c = 0.167(13)T_F$ [29] we get $(\eta/n)_{T=T_c} \gtrsim 0.41$. There is some uncertainty in this result due to higher order loop corrections, and due to the uncertainty in the cutoff. We note, however, that the bound scales as $\Lambda^{1/3}$, so that a factor 2 uncertainty in Λ only translates into a 25% uncertainty in the bound. As a conservative estimate we take $\Lambda \simeq 0.5K_{hyd}$ and

$$\left(\frac{\eta}{n}\right)_{T=T_c} \gtrsim 0.32. \quad (34)$$

We can illustrate this result further by using the leading order kinetic theory result as an estimate for the bare viscosity. This is consistent because kinetic theory takes into account

¹ We note that the lower bound on η/s is attained at a non-zero temperature. In particular, $(\eta/s)_{phys} \gtrsim 1.005(n/s)(T/T_F)^{1/2}(P/(nT))^{1/6}$ is minimal at a temperature $T \sim T_c$ because $(s/n) \sim T^3$ for $T \ll T_c$ and $(s/n) \sim \log(T)$ for $T \gg T_c$. Using the entropy per particle measured in [29] and $\lambda = 0.5K_{hyd}$, we obtain a fairly broad minimum $(\eta/s)_{phys} \gtrsim 0.3$ in the regime $T/T_F \sim (0.3 - 0.8)$.

effects at distances $l \lesssim l_{mfp}$ but, unless additional resummation is performed, it does not include fluctuations at length scales $l \gtrsim l_{mfp}$. The kinetic result is [30]

$$\eta = \frac{15}{32\sqrt{\pi}} (mT)^{3/2}. \quad (35)$$

In Fig. 2 we show the bare viscosity and the physical viscosity including the effects of fluctuations. We observe that the viscosity has a minimum $\eta/n \simeq 0.5$ at a temperature $T \simeq 0.2T_F$.

At the phase transition the entropy per particle is $s/n \simeq 0.8$ and equ. (34) translates into $(\eta/s)_{T_c} \gtrsim 0.4$. This number is comparable to experimental results reported by Cao et al. [6] and to the T-matrix calculation of Enss et al. [31], but larger than the Path Integral Monte Carlo (PIMC) results obtained by Wlazlowski et al. [32]. These authors find $(\eta/s)_{min} \gtrsim 0.2$, quite close to the AdS/CFT bound $(\eta/s) \geq 1/(4\pi)$. If the low value of η/s obtained in the PIMC calculation is indeed correct then the breakdown scale of hydrodynamics would have to be much smaller than the estimate given in equ. (30). Another possibility is that for the lattice spacing used in [32] one cannot resolve non-analytic contributions to the spectral function. Equ. (28) implies that for small ω

$$\eta(\omega) = \eta - \sqrt{\omega} T \frac{7 + \left(\frac{3}{2}\right)^{3/2}}{240\pi D_\eta^{3/2}}. \quad (36)$$

This relation shows that for small values of η it is difficult to resolve the zero frequency limit of the shear viscosity. The width of the non-analytic structure in the spectral function can be estimated by assuming that the fluctuation term in equ. (29) is related to the $\sqrt{\omega}$ term in the spectral function. This assumption leads to $\Delta\omega \simeq 0.3T(n/\eta)$, where we have used $\Lambda \simeq 0.5K_{hyd}$.

V. TWO DIMENSIONAL SYSTEMS

It is interesting to consider the role of fluctuations in two dimensional systems. The results in Sect. III are easily generalized to two spatial dimensions. Aside from the obvious substitution $d^3k/(2\pi)^3 \rightarrow d^2k/(2\pi)^2$ the only difference is that in two spatial dimensions the shear contribution to the sound attenuation constant is $\Gamma = D_\eta$ instead of $\Gamma = \frac{4}{3}D_\eta$. In both two and three dimensions the dominant contribution to G_R^{xyxy} arises from the one loop

diagram involving either a pair of shear modes or a pair of sound modes. In $d = 2$ the loop integral is logarithmically divergent and

$$G_R^{xyxy}(\omega, 0)|_{tot} = const - i\omega \frac{T}{16\pi D_\eta} \left[\log \left(\frac{\sqrt{2} D_\eta \Lambda^2}{\omega} \right) + i\frac{\pi}{2} \right]. \quad (37)$$

The imaginary part can be interpreted as a correction to the shear viscosity. We find

$$\eta_{phys} = \eta + \frac{mT}{16\pi} \frac{n}{\eta} \log \left(\frac{\sqrt{2} \Lambda^2}{m\omega} \frac{\eta}{n} \right) \quad (38)$$

which diverges logarithmically as $\omega \rightarrow 0$. This divergence is well known [33–35], and it has been observed in molecular dynamics and lattice gas simulations [36, 37]. To the best of our knowledge it has not been observed experimentally. Equ. (38) shows that the effect is large in systems that have a small value of the bare shear viscosity.

The shear viscosity of a dilute two-dimensional Fermi gas was recently computed in [38, 39]. The result is

$$\eta = \frac{mT}{2\pi^2} \left(\left[\log \left(\frac{5T}{2T_{a,2d}} \right) \right]^2 + \pi^2 \right), \quad (39)$$

where $T_{a,2d} = 1/(ma_{2d})^2$ and a_{2d} is the scattering length in two dimensions [40]. In two dimensions there is no scale invariant fluid except in the non-interacting limit $a_{2d} \rightarrow 0$. The most strongly correlated fluid corresponds to $T_{a,2d} \simeq T_F$, which implies that the dimer binding energy is equal to the Fermi energy. The viscosity of the two dimensional gas was recently studied by measuring the damping of the quadrupole mode [41] in a harmonic potential. The frequency of this mode is $\omega = \sqrt{2}\omega_\perp$, where ω_\perp is the two dimensional oscillator frequency. The confinement frequency sets the scale for the Fermi temperature of the trap, $T_F^{trap} = N^{1/2}\omega_\perp$. We can use these relations to translate the frequency dependence of the shear viscosity into the dependence on the number of particles. In Fig. 2 we show the kinetic theory result as well as the viscosity with fluctuations included for two different values of the particle number. The main dependence on N is of the form $(\eta/n)_{phys} \sim \frac{1}{16\pi} \log(N)$. We observe that fluctuations make a significant contribution to the shear viscosity, but the logarithmic divergence with N is fairly slow, and one will need significantly larger numbers of particles than what is available in current experiments ($N \simeq 4 \cdot 10^3$ in [41]) to see the effect clearly.

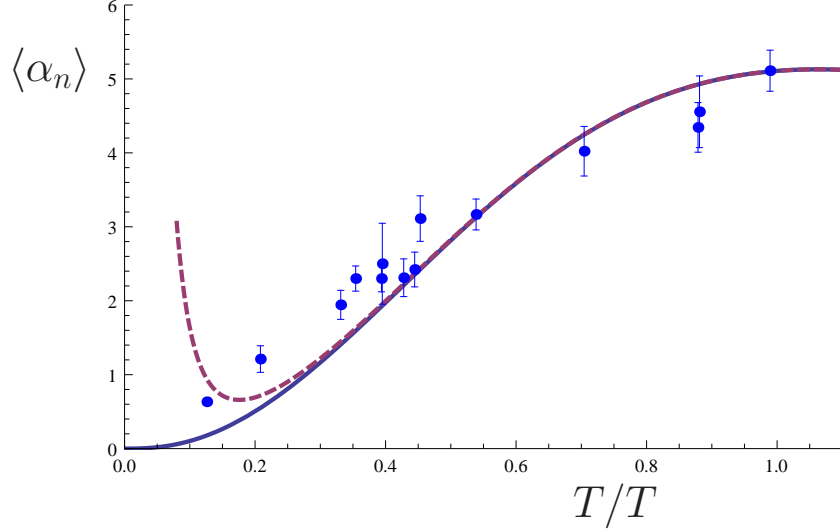


FIG. 3: Trap averaged shear viscosity to density ratio $\langle \alpha_n \rangle$. We show $\langle \alpha_n \rangle$ as a function of T/T_F^{trap} , where $T_F^{trap} = (3\lambda N)^{1/3}\omega_\perp$ is the Fermi temperature of the trap. We have chosen $N = 2 \cdot 10^5$ and $\lambda = 0.045$ as in [43]. The solid line shows the kinetic theory result, the dashed line includes fluctuation corrections to the shear viscosity. The data are from [44], which is a reanalysis of the results reported in [43]. We do not show data in the superfluid regime $T \ll T_c$.

VI. TRAPPED ATOMIC GASES

In this section we will try to make contact with experiments that study the damping of collective modes in trapped Fermi gases. We are interested in the question whether it is possible to establish the role of hydrodynamic fluctuations by studying the scaling of the damping constant with temperature or particle number. A review of the hydrodynamic theory of collective modes can be found in [42].

Consider a trapped gas with N particles in a harmonic potential with trapping frequencies $\omega_x = \omega_y = \omega_\perp$ and $\omega_z = \lambda\omega_\perp$. In a typical experiment $N = (10^5 - 10^6)$ and $\lambda = (0.02 - 0.05)$ [43, 45–47]. The transverse breathing mode has a frequency $\omega = \sqrt{10/3}\omega_\perp$ and the damping constant is

$$\Gamma_{br} = \frac{\langle \alpha_n \rangle}{(3N\lambda)^{1/3}} \frac{\omega_\perp}{(E_0/[N\epsilon_F])}, \quad (40)$$

where E_0 is the total (potential and internal) energy of the trapped gas, $\epsilon_F = (3N\lambda)^{1/3}\omega_\perp$ is the Fermi energy of the trapped system, and $\langle \alpha_n \rangle = \frac{1}{N} \int d^3x \eta(x)$ is the trap average of

the shear viscosity. Taking into account relaxation time effects we have

$$\langle \alpha_n \rangle = \frac{1}{N} \int d^3x \frac{\eta(x)}{1 + \omega^2 \tau_R(x)^2}. \quad (41)$$

We take the bare shear viscosity from kinetic theory, equ. (35), and compute the physical viscosity from equ. (32). We note that the bare viscosity only depends on T , which is independent of the position in the trap. The fluctuation term is largest at the center of the trap. We also use kinetic theory to determine the bare relaxation time, $\tau_R = \eta/P$, and use equ. (28) to determine the physical relaxation time. This corresponds to

$$(\tau_R \eta)_{phys} = \tau_R \eta + \frac{\left[7 + \left(\frac{3}{2}\right)^{3/2}\right] T}{240 \pi \omega^{1/2}} \left(\frac{\rho}{\eta}\right)^{3/2}. \quad (42)$$

We note that the bare relaxation time is inversely proportional to the local pressure and depends on the position in the trap. In particular, $\tau_R \eta$ is large in the dilute part of the cloud. Fluctuations, on the other hand, increase the relaxation time near the center of the trap.

In the following we will use the high temperature approximation for the density of the cloud. This is consistent with using kinetic theory for the bare shear viscosity and relaxation time. It also provides a very accurate description of the tail of the density distribution at essentially all temperatures. We have

$$n(x) = N \left(\frac{m \bar{\omega}^2}{2\pi T} \right)^{3/2} \exp \left(- \sum_i \frac{m \omega_i^2 x_i^2}{2T} \right), \quad (43)$$

where $\bar{\omega} = (\omega_\perp^2 \omega_z)^{1/3}$. Results for $\langle \alpha_n \rangle$ as a function of T/T_F^{trap} with $T_F^{trap} = (3N\lambda)^{1/3} \omega_\perp$ are shown in Fig. 3. We have used $N = 2 \cdot 10^5$ and $\lambda = 0.045$ as in the experiment of Kinast et al. [43]. The solid line shows the result using kinetic theory for η and τ_R , and the dashed line includes the fluctuation term in η . We find that for the parameters considered here corrections to the relaxation time are very small.

We observe that kinetic theory describes the data for $T \gtrsim 0.4 T_F^{trap}$ well. Fluctuations are important for $T \lesssim 0.2 T_F^{trap}$, leading to a minimum in $\langle \alpha_n \rangle$. We note that the critical temperature is $T_c \simeq 0.2 T_F^{trap}$ [48], and we do not expect the theory used in this section, which is based on kinetic theory in the dilute limit, to reproduce experiment for $T \ll T_c$. It was recently suggested that the data in this regime are dominated by the transition from hydrodynamic to ballistic behavior [49].

Finally, we have looked at the role of fluctuations in the experiment of Vogt et al. [41]. In this experiment the damping of two dimensional quadrupole mode was measured for $N = 4 \cdot 10^3$. The dependence on the scattering length was studied for $\log(k_F a_{2d}) > 0$ at $T/T_F = 0.48$, and the temperature dependence was studied in the range $T/T_F^{trap} = (0.3-0.8)$ for $\log(k_F a_{2d}) = (2.7-42)$. We find that for this range of parameters the role of fluctuations is always small. Fluctuations lead to a significant enhancement of the damping constant if $\log(k_F a_{2d}) \sim 0$ and $N \gtrsim 10^5$. This enhancement grows as $\log(N)$, but the logarithmic growth is difficult to disentangle from a $\log(N)$ term related to the dilute corona, see [38].

VII. CONCLUSIONS AND OUTLOOK

We have studied the role of hydrodynamics fluctuations in the dilute Fermi gas. Our main findings are:

1. Hydrodynamic fluctuations lead to a minimum in the shear viscosity. The physical origin of the minimum is the contribution of shear and sound modes to momentum transport. If the bare viscosity is small and sound and shear modes are weakly damped and this contribution becomes large. The magnitude of the minimum shear viscosity is weakly sensitive to the cutoff scale for the hydrodynamic description. Allowing for a factor of two uncertainty in our estimate of Λ we find $\eta/n \gtrsim 0.3$. This bound is not rigorous, but it can only be violated if the domain of applicability of hydrodynamics is anomalously small. Our estimate is close to the trap averaged value of η/n at $T \simeq T_c$ measured experimentally [6], but larger than the minimum $\eta/n \simeq 0.15$ obtained in a recent lattice simulation [32]. We note that the bound itself is purely classical, \hbar only enters through thermodynamic quantities².
2. Fluctuations cause a $1/\sqrt{\omega}$ divergence of the viscous relaxation time in three dimensions, and a $\log(\omega)$ divergence of the shear viscosity in two dimensions. These effects are independent of the cutoff and only depend on the value of the bare shear viscosity.

² Consider the bound on η/s . Ignoring numerical factors we have $\eta/s \gtrsim (n/s)(mT/n^{2/3})^{1/2}(P/(nT))^{1/6}$. At large temperature the ratio n/s depends weakly on T and the bound grows as $T^{1/2}$. At low T the entropy per particle increases sharply when the system reaches quantum degeneracy, which corresponds to $mT \sim \hbar^2 n^{2/3}$. This implies that at the minimum $\eta/s \sim \hbar$.

The existence of non-analytic terms implies that, strictly speaking, the two dimensional Navier-Stokes equation as well as the three dimensional second order (Burnett) equations are not consistent unless fluctuating forces are taken into account. We note, however, that real flows that can be studied in experiment involve finite frequencies or time scales, and fluctuating forces may not be important.

3. We have studied the importance of fluctuations for the damping of trapped Fermi gases. The corrections are generally small in the range of parameters that have been experimentally investigated. A possible exception is the three dimensional unitary gas near T_c . In this case fluctuations may lead to enhanced damping³. Fluctuations lead to a $\log(N)$ divergence in the damping constant of the two dimensional Fermi gas, but this effect is difficult to observe unless the number of particles is varied by more than an order of magnitude.

There are a number of interesting formal questions that we have not studied in this paper. In order to study higher order corrections it is useful to start from an effective action for hydrodynamic fluctuations. This could be done using the methods developed in [16, 19]. The effective action might also be useful for studying the renormalization group evolution in more detail. In this work we have simply assumed that the bare shear viscosity can be computed in kinetic theory. It would be desirable to provide a more rigorous justification for this approximation by studying the matching between kinetic theory and hydrodynamics. Finally, in this work we have restricted ourselves to studying the effect of fluctuations on the damping of collective modes. In this case it is straightforward to take into account a frequency dependent shear viscosity and relaxation time. If one considers hydrodynamic flows that are not periodic in time, for example the elliptic flow experiment described in [6], one has to solve the hydrodynamic equations with fluctuating forces. This method has been studied in the context of microfluidic systems, see [50] and references therein.

We have shown that hydrodynamic fluctuations are important if the bare viscosity is small in the low temperature limit. The main physical question is whether this scenario is

³ Note that we have not considered the role of critical fluctuations. The superfluid transition is described by model F in the classification of Hohenberg and Halperin [4]. This model does not contain direct couplings between the order parameter and the momentum density, and the calculation discussed in our work is not directly affected by critical fluctuations.

realized in the unitary Fermi gas near T_c , or whether other effects, like pairing correlations, a pseudo-gap or phonons are more important [31, 51–53]. This question is probably difficult to address experimentally, but it can be studied by analyzing the spectral function of the stress tensor. For this purpose it is necessary to construct models of the spectral function that include fluctuations. These models can be confronted with quantum Monte Carlo data, for example the recent work of Wlazlowski et al. [32].

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Appendix A: Stress tensor correlation function in hydrodynamics

1. Low energy expansion

In this section we provide some additional details regarding the low energy expansion of the retarded stress tensor correlation function. Our starting point is the symmetrized correlation function

$$G_S^{xyxy}(\omega, \mathbf{k}) = \int d^3x \int dt e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \left\langle \frac{1}{2} \{ \Pi_{xy}(t, \mathbf{x}), \Pi_{xy}(0, 0) \} \right\rangle. \quad (\text{A1})$$

We use the expression for the stress tensor in hydrodynamics and expand in small fluctuations, $\delta\rho, \delta T, v_i$, and in the number of derivatives. We get

$$\Pi_{xy} = \rho_0 v_x v_y - \eta_0 (\nabla_x v_y + \nabla_y v_x) - \left[\left(\frac{\partial \eta}{\partial \rho} \right)_T \delta \rho + \left(\frac{\partial \eta}{\partial T} \right)_\rho \delta T \right] (\nabla_x v_y + \nabla_y v_x) + \dots \quad (\text{A2})$$

where we have dropped terms of order δ^3 and ∇^2 . The diagrammatic expansion can be derived by inserting equ. (A2) into equ. (A1) and factorizing the expectation value into pairs of fluctuating fields. The retarded correlation function is obtained by replacing one of the symmetrized functions by a retarded function. In Section III we computed the one loop diagram that arises from the first term in equ. (A2). This term has no spatial derivatives, and we find a contribution of the form

$$G_R^{xyxy}(\omega, 0) \sim T \Lambda^3 \left\{ 1 + c_1 \frac{\omega}{D_\eta \Lambda^2} + c_{3/2} \frac{\omega^{3/2}}{D_\eta^{3/2} \Lambda^3} + \dots \right\}, \quad (\text{A3})$$

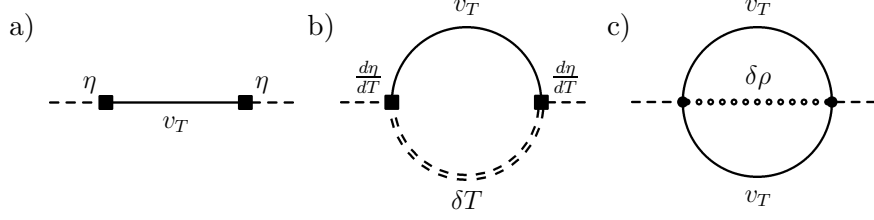


FIG. 4: Diagrammatic representation of higher order fluctuation contributions to the stress tensor correlation function. Solid lines labeled v_T denote the transverse velocity correlator, wavy lines labeled v_L denote the longitudinal velocity correlator. Temperature fluctuations are shown as double dashed lines, and density fluctuations are shown as dotted lines. Vertices shown as dots contain no derivatives, whereas vertices labeled by squares contain spatial derivatives.

where c_1 and $c_{3/2}$ are numerical constants. We observe that the low energy expansion involves powers of $\omega/(D_\eta \Lambda^2)$.

The second term contains spatial derivatives, and it gives rise to a tree diagram (Fig. 4a) which vanishes as $k \rightarrow 0$. The third term in equ. (A2) involves derivatives of the shear viscosity with respect to ρ and T . In kinetic theory $[(\partial\eta)/(\partial\rho)]_T$ vanishes at leading order in $n\lambda_{dB}^3$, where λ_{dB} is the de Broglie wave length. The dominant term therefore involves fluctuations of the temperature, $G_S \sim [(\partial\eta)/(\partial T)]^2 \langle \delta T \delta T \rangle \langle \nabla_x v_y \nabla_x v_y \rangle$ (plus permutations $x \rightarrow y$), see Fig. 4b. This is a one-loop graph with vertices that contain one spatial derivative. For non-zero external momenta this graph gives a contribution to G_R which is suppressed by k^2/K_{hyd}^2 relative to equ. (A3). For zero external momentum the diagram has power divergences which contribute at $O(\Lambda^2/K_{hyd}^2)$. This is not small if $\Lambda \sim K_{hyd}$, but power divergences can be absorbed into the transport coefficients⁴.

The leading no-derivative contribution of order δ^3 in the stress tensor is of the form $\Pi_{xy} = (\delta\rho)v_x v_y$. This term generates the two-loop diagram shown in Fig. 4c. The extra loop

⁴ It is well known that power counting is not manifest in effective field theories regularized by a momentum cutoff [55]. This problem can be circumvented using dimensional regularization (DR), which automatically eliminates all power divergent terms. In our context this is a disadvantage because we find that the leading one loop divergence in the stress tensor correlation function represents an important physical effect. The linear divergence is preserved in a modified version of dimensional regularization called “power divergence subtraction” (PDS) [56], which keeps the pole corresponding to the logarithmic divergence in two spatial dimensions.

integral involves three powers of momentum, and the extra propagator is proportional to $1/\rho$. As a consequence, the graph is suppressed by Λ^3/k_F^3 . This can be written as $(\Lambda/k_F)^3 \lesssim (K_{hyd}/k_F)^3 \simeq 1/(nl_{mfp}^3)$, where we have used the kinetic theory estimate $\eta \simeq n(mT)^{1/2}l_{mfp}$.

In summary, the hydrodynamic expansion of the correlation function $G_R^{xy}(\omega, \mathbf{k})$ involves powers of k/K_{hyd} , where $K_{hyd} = (P/\rho)^{1/2}D_\eta^{-1} \sim l_{mfp}^{-1}$ is the breakdown scale defined in Section IV. Since $G_R^{xy}(\omega, \mathbf{k})$ is diffusive the frequency scales as $\omega \sim D_\eta k^2$ and the frequency expansion involves $(\omega/[D_\eta K_{hyd}^2])$. Powers of k/K_{hyd} arise from higher derivative terms in the currents or from loop graphs. Loop graphs are additionally suppressed by $(K_{hyd}/k_F)^3 \simeq 1/(nl_{mfp}^3)$. Loop graphs are important because they lead to non-analytic effects, logarithms and fractional powers of ω and k^2 . For low viscosity fluids the mean free path is short, $nl_{mfp}^3 \sim 1$, and there is no suppression of loops relative to gradient terms.

2. Contact terms

The symmetrized correlation function contains a contact term which can be determined using the fluctuation-dissipation theorem [11, 57]

$$\left\langle \frac{1}{2} \{ \Pi_{xy}(t, \mathbf{x}), \Pi_{xy}(t', \mathbf{x}') \} \right\rangle = 2\eta T \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'). \quad (\text{A4})$$

In frequency space this gives the contribution of the bare shear viscosity to the retarded correlation function, $G_R = i\omega\eta$. The result therefore justifies combining the bare and loop corrections as in equ. (29). The contact term can also be obtained using the velocity correlation function combined with the conservation laws [17]. Momentum conservation implies $\nabla_i \Pi^{ij} = -\frac{\partial}{\partial t}(\rho v^j)$ and

$$k_x^2 G_S^{xyxy}(\omega, k_x) = \omega^2 \Delta_S^{yy}(\omega, k_x). \quad (\text{A5})$$

Using the explicit form of the velocity correlation function we find

$$G_S^{xyxy}(\omega, k_x) = 2\eta T \left\{ 1 - \frac{(D_\eta k_x^2)^2}{\omega^2 + (D_\eta k_x^2)^2} \right\}, \quad (\text{A6})$$

which contains the contact term $2\eta T$.

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